A conditioned local limit theorem for non-negative random matrices

M. Peigné, Thi da Cam Pham

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M. Peigné [(1)] & C. Pham [(2)],[(3)]
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Abstract

Let $(S_n)_n$ be the random process on $\mathbb{R}$ driven by the product of i.i.d. non-negative random matrices and τ its exit time from $]0, +\infty[$. By using the adapted strategy initiated by D. Denisov and V. Wachtel, we obtain an asymptotic estimate and bounds of the probability that the process $(S_k)_k$ remains non negative up to time n and simultaneously belongs to some compact set $[b, b + \ell] \subset \mathbb{R}^{*+}$ at time n.

Keywords: local limit theorem, random walk, product of random matrices, Markov chains, first exit time

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1 Introduction and main results

1.1 Motivation

Random walks conditioned to staying positive is a popular topic in probability. In addition to their own interest, such as information about the maxima and the minima, the ladder variables and the ladder epoch of random walks on $\mathbb{R}$, they are also important in view of their applications, for instance in queuing theory, in coding the genealogy of Galton-Watson trees or else as models for polymers and interfaces; we refer to [4] and references therein.

The first interesting question is to determine the asymptotic behavior of the exit time from the half line $[0, +\infty[$, and then to prove limit theorems for the process restricted to this half line or conditioned to remain there. More precisely, let $(S_n)_{n\geq 1}$ be a random walk in $\mathbb{R}$ whose increments are independent with common distribution. Assume that $(S_n)_{n\geq 1}$ is centered and let τ be its exit time from $[0, +\infty[$. Then, for any $a, b, \ell > 0$, as $n \to +\infty$,

$$\mathbb{P}_a(\tau > n, a + S_n \in [b, b + \ell[) \sim c\, \frac{h^+(a)h^-(b)}{n^{3/2}}\, \ell, \tag{1.1}$$

where c is a positive constant and h^+ and h^- are the renewal functions associated to $(S_n)_{n\geq 1}$, based on ascending and descending ladder heights (in particular these functions are positive). The increments being independent and identically distributed, the classical approach relies on the Wiener Hopf factorization and related identities associated with the names of Baxter, Pollaczek and Spitzer; important references in the field are given by Feller and Spitzer in their books [7], [25].

[1]Institut Denis Poisson UMR 7013, Université de Tours, Université d'Orléans, CNRS France. email: peigne@univ-tours.fr

[2]CERADE ESAIP, 18 rue du 8 mai 1945 - CS 80022 - 49180 St-Barthélemy d'Anjou ; email: dpham@esaip.org

[3]LAREMA UMR CNRS 6093, Angers, France.

Important conceptual difficulties arise both when the random walk $(S_n)_{n\geq 1}$ is $\mathbb{R}^d$-valued with $d \geq 2$ (the half line being replace by a general cone of the Euclidean space), or when the increments of the random walk are no longer independent. As far as we know, equivalent theory based on factorizations for these processes does not exist. In dimension $d \geq 2$, the Wiener-Hopf factorization method works when the cone is a half space but breaks down for more general cones. Any attempt to develop a theory of fluctuations for higher-dimensional random walks deals with the question: what would play the role of ladder epochs and ladder variables? [13]; Kingman showed in particular the impossibility of extending Baxter and Spitzer approach to random walks in higher dimension [16].

In 2055, D. Denisov and V. Wachtel developed a new approach to study the exit time from a cone of a random walk and several consequent limit theorems [5]. Their strategy, based on the approximation of these walks suitably normalized by a Brownian motion, with a strict control of the speed of convergence, is promising, powerful and flexible. It allows in particular to approach the random walks whose jumps are not i.i.d.

This flexible approach could be adapted to the quantity $S_n(x) := \ln |g_n \cdots g_1(x)|$, where $(g_k)_{k\geq 1}$ is a sequence of i.i.d. random matrices, x is a non nul vector in $\mathbb{R}^d$ and $|\cdot|$ is the L_1 norm in $\mathbb{R}^d$; this process falls within the general framework of Markov walks on $\mathbb{R}$ satisfying some spectral gap assumption. The behavior of the tail of the distribution of $\tau_{x,a} := \inf\{n \geq 1 : a + \ln |g_n \cdots g_1(x)| \leq 0\}$ is known for a few years when the random matrices are invertible or non-negative [12] [21]. This is extended by I. Grama, R. Lauvergnat & E. Le Page in [10] to the case of Markov walks, under a spectral gap assumption. Nevertheless, the question of a local limit theorem for $\ln |g_n \cdots g_1(x)|$ confined in a half line still resists. In [11] such a statement holds for conditioned Markov walks over a finite state space, in which case the dual driving Markov chain also satisfies nice spectral gap properties; unfortunately, such a property does not hold for product of random matrices since it is not realistic to assume that the random matrices M_n act projectively on a finite set.

1.2 Notations and assumptions

We endow $\mathbb{R}^d$ with the L_1 norm $|\cdot|$ defined by $|x| := \sum_{i=1}^{d} |x_i|$ for any column vector $x = (x_i)_{1\leq i\leq d}$.

Let $\mathcal{S}$ be the set of $d \times d$ matrices with positive entries. We endow $\mathcal{S}$ with the standard multiplication of matrices, then the set $\mathcal{S}$ is a semigroup. For any $g = (g(i,j))_{1\leq i,j\leq d} \in \mathcal{S}$, we define v, endow $|\cdot|$ a norm on $\mathcal{S}$ and define N as follows,

$$v(g) := \min_{1\leq j\leq d} \Big(\sum_{i=1}^{d} g(i,j)\Big); \quad |g| := \sum_{i,j=1}^{d} g(i,j) \quad \text{and} \quad N(g) := \max\left(\frac{1}{v(g)}, |g|\right).$$

Notice that $N(g) \geq 1$ for any $g \in \mathcal{S}$.

Let $\mathcal{C}$ be the cone of column vectors defined by $\mathcal{C} := \{x \in \mathbb{R}^d \mid \forall 1 \leq i \leq d, x_i \geq 0\}$ and $\mathbb{X}$ be the limited standard simplex defined by $\mathbb{X} := \{x \in \mathcal{C} \mid |x| = 1\}$. For any $x \in \mathcal{C}$, we denote by $\tilde{x}$ the corresponding row vector and set $\tilde{\mathcal{C}} = \{\tilde{x} \mid x \in \mathcal{C}\}$ and $\tilde{\mathbb{X}} = \{\tilde{x} \mid x \in \mathbb{X}\}$.

We consider the following actions:

- the linear action of $\mathcal{S}$ on $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$) defined by $(g, x) \mapsto gx$ (resp. $(g, \tilde{x}) \mapsto \tilde{x}g$) for any $g \in \mathcal{S}$ and $x \in \mathcal{C}$,

- the projective action of $\mathcal{S}$ on $\mathbb{X}$ (resp. $\tilde{\mathbb{X}}$) defined by $(g,x) \mapsto g\cdot x := \dfrac{gx}{|gx|}$ (resp. $(g,\tilde{x}) \mapsto \tilde{x}\cdot g = \dfrac{\tilde{x}g}{|\tilde{x}g|}$) for any $g \in \mathcal{S}$ and $x \in \mathbb{X}$.

It is noticeable that $0 < v(g)\ |x| \leq |gx| \leq |g|\ |x|$ for any $x \in \mathcal{C}$.

For any fixed $x \in \mathbb{X}$ and $a \geq 0$, we denote by $\tau_{x,a}$ the first time the random process $(a + \ln|g_n \cdots g_1 x|)_n$ becomes negative, i.e.

$$\tau_{x,a} := \min\{n \geq 1 :\ a + \ln|g_n \cdots g_1 x| \leq 0\}.$$

We impose the following assumptions on μ.

P1 *Moment assumption: There exists $\delta_1 > 0$ such that $\int_{\mathcal{S}} |\ln N(g)|^{2+\delta_1}\mu(dg) < +\infty$.*

Notice that hypothesis **P1** is weaker than the one in [21] where exponential moments are required; the argument developed in [21] is improved by allowing various parameters (see [20], *Proof of Theorem 1.6 (d)*).

P2 *Irreducibility assumption: There exists no affine subspaces A of $\mathbb{R}^d$ such that $A \cap \mathcal{C}$ is non-empty and bounded and invariant under the action of all elements of the support of μ.*

This assumption is classical in the context of product of positive random matrices, it ensures in particular that the central limit theorem satisfied by these products is meaningful since the variance is positive (see Corollary 3 in [14]).

P3 *There exists $B > 0$ such that for μ-almost all g in $\mathcal{S}$ and any $1 \leq i,j,k,l \leq d$*

$$\frac{g(i,j)}{B} \leq g(k,l) \leq B\ g(i,j).$$

This is a classical assumption for product of random matrices with positive entries, it was first introduced by H. Furstenberg and H. Kesten [8].

P4 *Centering The upper Lyapunov exponent γ_μ is equal to* 0.

P5 *There exists $\delta_5 > 0$ such that $\mu\{g \in \mathcal{S} : \forall x \in \mathbb{X}, \ln|gx| \geq \delta_5\} > 0$.*

Condition **P5** ensures that uniformly in $x \in \mathcal{C}\setminus\{0\}$, the probability that the process $(a + \ln|g_n \cdots g_1 x|)_{n\geq 1}$ remains in the half line $[0,+\infty[$ is positive. It is satisfied for instance when $\mu\{g \mid v(g) > 1\} > 0$.

As it is usual in studying local probabilities, one has to distinguish between "lattice" and "non lattice" cases. The "non lattice" assumption ensures that the $\mathbb{R}$-component of the trajectories of the Markov walk $(X_n, S_n)_{n\geq 0}$ do not live in the translation of a proper subgroup of $\mathbb{R}$; in the contrary case, when μ is lattice, a phenomenon of cyclic classes appears which involves some complications which are not interesting in our context. We refer to equality (2.1) in section 2 for a precise definition in the context of products of random matrices.

P6 *Non-lattice assumption The measure μ is non-lattice.*

The tail of the distribution of $\tau_{x,a}$ has been the subject of an extensive study in [21]: under hypotheses **P1-P5**, there exists a positive Borel function $V : \mathbb{X}\times\mathbb{R}^+ \to \mathbb{R}^+$ such that as

$n \to +\infty$,

$$\mathbb{P}(\tau_{x,a} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} V(x,a).$$

In the sequel, we also need to consider the process $(b - \ln|\tilde{x} g_1 \cdots g_n|)_n, \tilde{x} \in \tilde{\mathbb{X}}, b \in \mathbb{R}^+$, associated to the right products $g_1 \dots g_n, n \geq 1$. We thus also consider the stopping time

$$\tilde{\tau}_{\tilde{x},b} := \min\{n \geq 1 : \ b - \ln|\tilde{x} g_1 \cdots g_n| \leq 0\}.$$

As above, there exists a positive Borel function $\tilde{V} : \mathbb{X} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that as $n \to +\infty$,

$$\mathbb{P}(\tilde{\tau}_{\tilde{x},b} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} \tilde{V}(\tilde{x},b).$$

At last, as $n \to +\infty$, the sequence $\left(\dfrac{a + \ln|g_n \cdots g_1 x|}{\sigma\sqrt{n}}\right)_n$ conditioned to the event $(\tau_{x,a} > n)$ converges weakly towards the Rayleigh distribution on $\mathbb{R}^+$ whose density equals $y\ e^{-y^2/2} 1_{\mathbb{R}^+}(y)$. Properties of the function V are precisely stated in section 2.

The natural question is to get a local limit theorem for the process $(a + \ln|g_n \cdots g_1 x|)_{n \geq 1}$ forced to stay positive up to time n, in other words to describe the behavior of the quantity $\mathbb{P}(\tau_{x,a} > n, a + \ln|g_n \cdots g_1 x| \in [b, b+\ell])$ as $n \to +\infty$, when $a, b > 0$ and $\ell > 0$.

1.3 Main statements

We first state a version of the Gnedenko local limit theorem.

Theorem 1.1 *Assume hypotheses* **P1-P6**. *Then, as $n \to +\infty$, for any $x \in \mathbb{X}, a \in \mathbb{R}$, any $b \geq 0$ and $\ell > 0$,*

$$\lim_{n \to +\infty} \left| n\mathbb{P}(\tau_{x,a} > n, a + \ln|g_n \cdots g_1 x| \in [b, b+\ell]) - \frac{2\sqrt{2\pi}}{\sigma^2\sqrt{n}} V(x,a)\ b\ e^{-b^2/2n}\ \ell \right| = 0,$$

the convergence being uniform in $x \in \mathbb{X}$ and $b \geq 0$.

Notice that Theorem 1.1 says only that this probability is $o(n^{-1})$. The following theorem describes an asymptotic behavior of $\mathbb{P}(\tau_{x,a} > n, a + \ln|g_n \cdots g_1 x| \in [b, b+\ell])$. Recall that $\Delta = \ln\delta$ where δ is defined in Lemma 2.2.

Theorem 1.2 *Assume hypotheses* **P1-P6**. *There exists positive constant $c, C > 0$ such that, for any $x \in \mathbb{X}, a, b \geq 0$ and $\ell > 0$,*

$$n^{3/2}\mathbb{P}(\tau_{x,a} > n, a + \ln|g_n \cdots g_1 x| \in [b, b+\ell]) \leq C\ V(x,a)\ \tilde{V}(x,b)\ \ell. \tag{1.2}$$

Furthermore, there exists $\ell_0, \Delta > 0$ such that, for $\ell > \ell_0$ and $b \geq \Delta$,

$$\liminf_{n \to +\infty} n^{3/2}\mathbb{P}(\tau_{x,a} > n, a + \ln|g_n \cdots g_1 x| \in [b, b+\ell]) \geq c\ V(x,a)\ \tilde{V}(x,b)\ \ell. \tag{1.3}$$

As for random walks with i.i.d. increments, it is expected that this probability is in fact equivalent to $n^{3/2}$ up to a positive constant. The argument relies on a combination of what is sometimes called "reverse time" and "duality" in the classical theory of random walks; roughly speaking, it relies on the fact that, for a classical random walk $(S_n)_{n\geq 1}$ with i.i.d. increments, the vectors $(S_1, S_2, \ldots, S_n)$ and $(S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n)$ have the same distribution. In [10], this idea has been developed in the context of Markov walks over a Markov chain with finite state space, it is technically much more difficult and so far, it escapes from the framework of random matrix products (see the paragraph before Lemma 2.3 for more detailed explanations). In the case of non-negative random matrices, the difference between $\ln|g_n \ldots g_1 x|$ and $\ln|g_n \ldots g_1|$ is uniformly bounded (see Lemma 2.2 below), one can thus avoid the precise study of the associated dual chain[4] to obtain the above result, a bit less precise but still worth of interest.

Notation. Let c be a strictly positive constant and ϕ, ψ be two functions of some variable x; we denote by $\phi \overset{c}{\preceq} \psi$ (or simply $\phi \preceq \psi$) when $\phi(x) \leq c\ \psi(x)$ for any value of x. The notation $\phi \overset{c}{\asymp} \psi$ (or simply $\phi \asymp \psi$) means $\phi \overset{c}{\preceq} \psi \overset{c}{\preceq} \phi$.

2 Preliminaries

2.1 The killed Markov walk on $\mathbb{X} \times \mathbb{R}$ and its harmonic function

We consider a sequence of i.i.d. $\mathcal{S}$-valued matrices $(g_n)_{n\geq 0}$ with common distribution μ and denote the left and right product of matrices $L_{n,k} := g_n \ldots g_k$ and $R_{k,n} = g_k \cdots g_n$ for any $n \geq k \geq 0$.

We fix a $\mathbb{X}$-valued random variable X_0 and consider the Markov chain $(X_n)_{n\geq 0}$ defined by $X_n^{X_0} := L_{n,1} \cdot X_0$ for any $n \geq 1$; when $X_0 = x$, we set for simplicity $X_n = X_n^x$. Similarly, the $\tilde{\mathbb{X}}$-valued Markov chain $(\tilde{X}_n)_{n\geq 0}$ is defined by $\tilde{X}_n := \tilde{X}_0 \cdot R_{1,n}$ for any $n \geq 1$, where $\tilde{X}_0$ is a fixed $\tilde{\mathbb{X}}$-valued random variable.

Notice that the sequence $(g_{n+1}, X_n^x)_{n\geq 0}$ (resp. $(g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}$) is a $\mathcal{S} \times \mathbb{X}$ valued (resp. $\mathcal{S} \times \tilde{\mathbb{X}}$ valued) Markov chain with initial distribution $\mu \otimes \delta_x$ (resp. $\mu \otimes \delta_{\tilde{x}}$). Their respective transition probability P and Q are defined by: for any $(g, x) \in \mathcal{S} \times \mathbb{X}$ and any bounded Borel function $\varphi : \mathcal{S} \times \mathbb{X} \to \mathbb{C}, \phi : \mathcal{S} \times \tilde{\mathbb{X}} \to \mathbb{C}$,

$$P\varphi(g, x) := \int_{\mathcal{S}} \varphi(h, g \cdot x)\mu(\mathrm{d}h) \quad \text{and} \quad Q\phi(g, \tilde{x}) := \int_{\mathcal{S}} \varphi(h, \tilde{x} \cdot g)\mu(\mathrm{d}h).$$

We denote by $(\Omega = (\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}, \mathcal{F} = \mathcal{B}(\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}, (g_{n+1}, X_n^x)_{n\geq 0}, \theta, \mathbb{P}_x)$ the canonical probability space associated with $(g_{n+1}, X_n^x)_{n\geq 0}$, where θ is the classical "shift operator" on $(\mathcal{S} \times \mathbb{X})^{\otimes \mathbb{N}}$. Similarly $(\tilde{\Omega}, \tilde{\mathcal{F}}, (g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}, \tilde{\theta}, \mathbb{P}_{\tilde{x}})$ denotes the canonical probability space associated with $(g_{n+1}, \tilde{X}_n^{\tilde{x}})_{n\geq 0}$.

We introduce next the functions ρ ant $\tilde{\rho}$ defined for any $g \in \mathcal{S}$ and $x \in \mathbb{X}$ by

$$\rho(g, x) := \ln|gx| \quad \text{and} \quad \tilde{\rho}(g, \tilde{x}) := \ln|\tilde{x}g|.$$

Notice that $gx = e^{\rho(g,x)} g \cdot x$ and that ρ satisfies the "cocycle property":

$$\rho(gh, x) = \rho(g, h \cdot x) + \rho(h, x), \quad \forall g, h \in \mathcal{S} \text{ and } x \in \mathbb{X}.$$

[4]This study would require restrictive conditions on μ, for example the existence of a density.

This yields to the following basic decomposition

$$\ln |L_{n,1}x| = \sum_{k=0}^{n-1} \rho(g_{k+1}, X_k^x) \quad \text{and} \quad \ln |\tilde{x} R_{1,n}| = \sum_{k=0}^{n-1} \rho(g_{k+1}, \tilde{X}_k^{\tilde{x}}).$$

This is thus natural to introduce the following Markov walks on $\mathbb{X} \times \mathbb{R}$ and $\tilde{\mathbb{X}} \times \mathbb{R}$ defined by $S_n = S_0 + \sum_{k=0}^{n-1} \rho(g_{k+1}, X_k^x)$ and $\tilde{S}_n = \tilde{S}_0 - \sum_{k=0}^{n-1} \tilde{\rho}(g_{k+1}, X_k^{\tilde{x}})$ where S_0 and $\tilde{S}_0$ are real valued random variables. Notice that the sequences $(X_n, S_n)_{n\geq 0}$ and $(\tilde{X}_n, \tilde{\mathcal{S}}_n)_{n\geq 0}$ are Markov chains on $\mathbb{X} \times \mathbb{R}$ and $\tilde{\mathbb{X}} \times \mathbb{R}$ respectively, with transition probability $\widetilde{P}$ and $\widetilde{Q}$ defined by: for any $(x, a) \in \mathbb{X} \times \mathbb{R}$ and any bounded Borel functions $\Phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}, \Psi : \tilde{\mathbb{X}} \times \mathbb{R} \to \mathbb{C}$,

$$\widetilde{P}\Phi(x, a) = \int_{\mathcal{S}} \Phi(g \cdot x, a + \rho(g, x))\mu(\mathrm{d}g) \quad \text{and} \quad \widetilde{Q}\Psi(\tilde{x}, a) = \int_{\mathcal{S}} \Psi(\tilde{x} \cdot g, a - \tilde{\rho}(g, \tilde{x}))\mu(\mathrm{d}g).$$

For any $(x, a) \in \mathbb{X} \times \mathbb{R}$, we denote by $\mathbb{P}_{x,a}$ the probability measure on $(\Omega, \mathcal{F})$ conditioned to the event $(X_0 = x, S_0 = a)$ and by $\mathbb{E}_{x,a}$ the corresponding expectation; for simplicity, we set $\mathbb{P}_{x,0} = \mathbb{P}_x$ and $\mathbb{E}_{x,0} = \mathbb{E}_x$.

Hence for any $n \geq 1$,

$$\widetilde{P}^n\Phi(x, a) = \mathbb{E}[\Phi(L_{n,1} \cdot x, a + \ln |L_{n,1}x|)] = \mathbb{E}_{x,a}[\Phi(X_n, S_n)].$$

Next we consider the restriction $\widetilde{P}_+$ to $\mathbb{X} \times \mathbb{R}^+$ of $\widetilde{P}$ defined for any $(x, a) \in \mathbb{X} \times \mathbb{R}$ by:

$$\widetilde{P}_+\Phi(x, a) = \widetilde{P}(\Phi \mathbf{1}_{\mathbb{X}\times\mathbb{R}^+})(x, a).$$

Let us emphasize that $\widetilde{P}_+$ may not be a Markov kernel on $\mathbb{X}\times\mathbb{R}^+$. Furthermore, if $\tau := \min\{n \geq 1 : S_n \leq 0\}$ is the first time the random process $(S_n)_{n\geq 1}$ becomes non-positive, it holds for any $(x, a) \in \mathbb{X} \times \mathbb{R}^+$ and any bounded Borel function $\Phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$,

$$\widetilde{P}_+\Phi(x, a) = \mathbb{E}_{x,a}[\Phi(X_1, S_1); \tau > 1] = \mathbb{E}[\Phi(g_1 \cdot x, a + \ln |g_1 \cdot x|), a + \ln |g_1 \cdot x| > 0].$$

A positive $\widetilde{P}_+$-harmonic function V is any function from $\mathbb{X} \times \mathbb{R}^+$ to $\mathbb{R}^+$ satisfying $\widetilde{P}_+V = V$. We extend V by setting $V(x, a) = 0$ for $(x, a) \in \mathbb{X} \times \mathbb{R}^-_*$. In other words, the function V is $\widetilde{P}_+$-harmonic if and only if for any $x \in \mathbb{X}$ and $a \geq 0$,

$$V(x, a) = \mathbb{E}_{x,a}[V(X_1, S_1); \tau > 1].$$

Similarly, let $\tilde{\tau} := \min\{n \geq 1 : \tilde{S}_n \leq 0\}$ be the first time the random process $(\tilde{S}_n)_{n\geq 1}$ becomes non-positive; for any $(x, b) \in \tilde{\mathbb{X}} \times \mathbb{R}^+$ and any bounded Borel function $\Psi : \tilde{\mathbb{X}} \times \mathbb{R} \to \mathbb{C}$,

$$\mathbb{E}_{\tilde{x},b}[\Psi(\tilde{X}_1, \tilde{S}_1); \tilde{\tau} > 1] = \mathbb{E}[\Psi(\tilde{x} \cdot g_1, b - \ln |\tilde{x} \cdot g_1|); b - \ln |\tilde{x} \cdot g_1| > 0].$$

From Theorem II.1 in [14], under conditions P1-P3 introduced below, there exists a unique probability measure ν on $\mathbb{X}$ such that for any bounded Borel function φ from $\mathbb{X}$ to $\mathbb{R}$,

$$(\mu * \nu)(\varphi) = \int_{\mathcal{S}} \int_{\mathbb{X}} \varphi(g \cdot x)\nu(dx)\mu(dg) = \int_{\mathbb{X}} \varphi(x)\nu(dx) = \nu(\varphi).$$

Such a measure is said to be μ-invariant. When $\mathbb{E}[|\ln|A_1||] < +\infty$, the upper Lyapunov exponent associated with μ is finite and is expressed by

$$\gamma_\mu = \int_{\mathcal{S}} \int_{\mathbb{X}} \rho(g,x)\nu(dx)\mu(dg).$$

We are now able to give a precise definition of a lattice distribution μ. We say that the measure μ is *lattice* if there exists $t > 0, \epsilon \in [0, 2\pi[$ and a function $\psi : \mathbb{X} \to \mathbb{R}$ such that

$$\forall g \in T_\mu, \forall x \in \psi(T_\mu), \quad \exp\{it\rho(g,x) - i\epsilon + i(\psi(g \cdot x) - \psi(x))\} = 1, \tag{2.1}$$

where T_μ is the closed sub-semigroup generated by the support of μ.

It is also noticeable that the process $(X_n, S_n)_n$ is a semi-markovian random walk on $\mathbb{X} \times \mathbb{R}$ with the strictly positive variance $\sigma^2 := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}_x[S_n^2]$, for any $x \in \mathbb{X}$. Condition **P2** implies that $\sigma^2 > 0$; we refer to Theorem 5 in [14].

In [21], we establish the asymptotic behaviour of $\mathbb{P}(\tau_{x,a} > n)$ by studying the $\widetilde{P}_+$-harmonic function V. Firstly, we prove the existence of a $\widetilde{P}_+$-harmonic function properties.

Proposition 2.1 *Assume hypotheses* **P1-P5**. *Then there exists a $\widetilde{P}_+$-harmonic Borel function $V : \mathbb{X} \times \mathbb{R}^+ \to \mathbb{R}^+$, such that $t \mapsto V(x,t)$ is increasing on $\mathbb{R}^+$ for any $x \in \mathbb{X}$ and satisfies the following properties: there exist $c, C > 0$ and $A > 0$ such that for any $x \in \mathbb{X}$ and $a \geq 0$,*

$$c \vee (a - A) \leq V(x,a) \leq C\ (1 + a) \quad \text{and} \quad \lim_{a\to+\infty} \frac{V(x,a)}{a} = 1.$$

Furthermore, for any $x \in \mathbb{X}, a \geq 0$ and $n \geq 1$,

$$\sqrt{n}\mathbb{P}(\tau_{x,a} > n) \leq C\ V(x,a)$$

and as $n \to +\infty$,

$$\mathbb{P}(\tau_{x,a} > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} V(x,a).$$

At last, as $n \to +\infty$, the sequence $\left(\frac{a + \ln|L_{n,1}x|}{\sigma\sqrt{n}}\right)_n$ conditioned to $(\tau_{x,a} > n)$ converges weakly towards the Rayleigh distribution on $\mathbb{R}^+$ whose density equals $y\ e^{-y^2/2}1_{\mathbb{R}^+}(y)$, relatively to $\mathbb{P}_{x,a}$ for any $x \in \mathbb{X}$ and $a > 0$.

2.2 Product of positive random matrices in $\mathcal{S}_\delta$

For any fixed $B > 1$, let $\mathcal{S}_B$ denote the subset of $\mathcal{S}$ that includes matrices satisfying **P3**. Products of random matrices are first studied by H. Furstenberg and H. Kesten [8] for matrices in $\mathcal{S}_B$ and then being extended to elements of $\mathcal{S}$ by several authors (see [14] and references therein). The restrictive condition of H. Furstenberg and H. Kesten considerably simplifies the study. The following statement (see [8] Lemma 2) is a key argument in the sequel to control the asymptotic behaviour of the norm of products of matrices of $\mathcal{S}_B$. Let $T_{\mathcal{S}_B}$ be the semi-group generated by the set $\mathcal{S}_B$.

Lemma 2.2 *For any $g \in T_{\mathcal{S}_B}$ and $1 \le i,j,k,l \le p$,*

$$g(i,j) \overset{B^2}{\asymp} g(k,l). \tag{2.2}$$

In particular, there exist $\delta > 1$ such that for any $g,h \in T_{\mathcal{S}_B}$ and $x \in \mathbb{X}, \tilde{y} \in \tilde{\mathbb{X}}$,

1. $|gx| \overset{\delta}{\asymp} |g|$ *and* $|\tilde{y}g| \overset{\delta}{\asymp} |g|$,
2. $|\widetilde{y}gx| \overset{\delta}{\asymp} |g|$,
3. $|g||h| \overset{\delta}{\preceq} |gh| \le |g||h|$.

As a direct consequence, the sequence $(\ln|L_{n,1}x| - \ln|L_{n,1}|)_{n\ge 0}$ is bounded uniformly in $x \in \mathbb{X}$. This property is crucial in the sequel in order to apply the "reverse time" trick, an essential argument in the proofs of our main results.

When studying fluctuations of random walks $(S_n)_{n\ge1}$ with i.i.d. increments Y_k on $\mathbb{R}^d, d \ge 1$, it is useful to "reverse time" as follows. For any $1 \le k \le n$, the random variables $S_n - S_k = Y_{k+1} + \ldots + Y_n$ and $S_{n-k} = Y_1 + \ldots + Y_{n-k}$ have the same distribution. In the case of products of random matrices, the cocycle property $S_n(x) = \ln|L_{n,1}(x)| = S_k(x) + S_{n-k}(X_k)$ is more subtle and the same argument cannot be applied directly. The fact that the g_k belong to $\mathcal{S}_\delta$ comes to our rescue here, but the price to pay is the appearance of the constant $\Delta = \ln\delta$ that disturbs the estimates as follows. Up to this constant Δ, we can compare the distribution of $S_n(x) - S_k(x) =: S_{n-k}(X_k)$ to the one of $\ln|g_{k+1}\ldots g_n|$, then to the one of $\ln|g_1\ldots g_{n-k}|$ and at last to the one of $\ln|\tilde{y}g_1\ldots g_{n-k}| =: -\tilde{S}_{n-k}(y)$, for any $x,y \in \mathbb{X}$ (notice here that for this last quantity, the non-commutativity of the product of matrices forces us to consider the right linear action of the matrices $R_{1,n-k}$). It is the strategy that we apply repeatedly to obtain the following result.
Recall that $\Delta = \ln\delta$ where $\delta > 1$ is the constant which appears in Lemma 2.2.

Lemma 2.3 *For any $x,y \in \mathbb{X}, a,b \ge 0$ and $\ell > 0$,*

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \le \mathbb{P}_{\tilde{y},\ b+\ell+\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a, a+\ell+2\Delta]). \tag{2.3}$$

Similarly, for $a \ge \ell > 2\Delta > 0$ and $b \ge \Delta$,

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \ge \mathbb{P}_{\tilde{y},\ b-\Delta}(\tilde{\tau} > n, \tilde{S}_n \in [a-\ell, a-2\Delta]). \tag{2.4}$$

Proof. We begin with the demonstration of (2.3). For any $n \in \mathbb{N}, b > 0$ and $h > 0$, it follows that

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) \\
&\qquad = \mathbb{P}_x(a + S_1 > 0, \ldots, a + S_{n-1} > 0, a + S_n \in [b, b+\ell]) \\
&\qquad = \mathbb{P}_x(a + S_n - S_{n-1}\circ\theta > 0, \ldots, a + S_n - S_1\circ\theta^{n-1} > 0, a + S_n \in [b, b+\ell]) \\
&\qquad \le \mathbb{P}_x(b + \ell - S_{n-1}\circ\theta > 0, \ldots, b + \ell - S_1\circ\theta^{n-1} > 0, b + \ell - S_n \in [a, a+\ell]),
\end{aligned}$$

where θ is the shift operator and $S_{n-k}\circ\theta^k = \ln|L_{n,k+1}X_k^x|$ $\quad\mathbb{P}_x$-a.s. for any $0 \le k \le n-1$. By Lemma 2.2, the quantities $\ln|L_{n,k+1}X_k^x|$ and $\ln|\tilde{y}L_{n,k+1}|$ both belong to the interval

$[\ln|L_{n,k+1}|-\Delta, \ln|L_{n,k+1}|]$ for any $\tilde{y} \in \tilde{\mathbb{X}}$ and $0 \le k \le n-1$. Therefore $S_{n-k}\circ\theta^k \in [\ln|\tilde{y}L_{n,k+1}|-\Delta; \ln|\tilde{y}L_{n,k+1}|+\Delta]$ and as a result

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell])\\
&\quad\le\ \mathbb{P}(b+\ell+\Delta-\ln|\tilde{y}L_{n,2}|>0,\dots,b+\ell+\Delta-\ln|\tilde{y}L_{n,n}|>0,\\
&\qquad\qquad b+\ell+\Delta-\ln|\tilde{y}L_{n,1}|\in[a,a+\ell+2\Delta])\\
&\quad=\ \mathbb{P}(b+\ell+\Delta-\ln|\tilde{y}R_{1,n-1}|>0,\dots,b+\ell+\Delta-\ln|\tilde{y}R_{1,1}|>0,\\
&\qquad\qquad b+\ell+\Delta-\ln|\tilde{y}R_{1,n}|\in[a,a+\ell+2\Delta])
\end{aligned}$$

by using the fact that $(g_1,\dots,g_n)$ and $(g_n,\dots,g_1)$ have the same distribution

$$= \mathbb{P}_{\tilde{y},\ b+\ell+\Delta}(\tilde{\tau}>n, \tilde{S}_n\in[a,a+\ell+2\Delta]).$$

Similarly, for $a>\ell>2\Delta>0$ and $b>0$, we obtain the proof of (2.4) as follows.

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell])\\
&\quad=\mathbb{P}_x(a+S_1>0,\dots,a+S_{n-1}>0,a+S_n\in[b,b+\ell])\\
&\quad=\mathbb{P}_x(a+S_n-S_{n-1}\circ\theta>0,\dots,a+S_n-S_1\circ\theta^{n-1}>0,b\le a+S_n\le b+\ell)\\
&\quad\ge\mathbb{P}_x(b-S_{n-1}\circ\theta>0,\dots,b-S_1\circ\theta^{n-1}>0,a-\ell\le b-S_n\le a)\\
&\quad\ge\mathbb{P}(b-\Delta-\ln|\tilde{y}L_{n,2}|>0,\dots,b-\Delta-\ln|\tilde{y}L_{n,n}|>0,\\
&\qquad\qquad a-\ell\le b-\Delta-\ln|\tilde{y}L_{n,1}|\le a-2\Delta)\\
&\quad=\mathbb{P}(b-\Delta-\ln|\tilde{y}R_{1,n-1}|>0,\dots,b-\Delta-\ln|\tilde{y}R_{1,1}|>0,\\
&\qquad\qquad b-\Delta-\ln|\tilde{y}R_{1,n}|\in[a-\ell,a-2\Delta])\\
&\quad=\mathbb{P}_{\tilde{y},\ b-\Delta}(\tilde{S}_1>0,\dots,\tilde{S}_{n-1}>0,\tilde{S}_n\in[a-\ell,a-2\Delta])\\
&\quad=\mathbb{P}_{\tilde{y},\ b-\Delta}(\tilde{\tau}>n,\tilde{S}_n\in[a-\ell,a-2\Delta]).
\end{aligned}$$

Since $a>\ell>2\Delta>0$, the interval $[a-\ell,a-2\Delta]$ is not empty.

□

2.3 Limit theorem for product of positive random matrices

In this section, we state some classical results and preparatory lemmas, useful for the demonstration of Theorem 1.1 and Theorem 1.2. The following result plays a crucial role in this article.

Theorem 2.4 *([2], Theorem 3.2.2) Assume hypotheses* **P1-P6** *hold. Then for any continuous function $u:\mathbb{X}\to\mathbb{R}$ and any continuous function with compact support $\varphi:\mathbb{R}\to\mathbb{R}$, it holds*

$$\lim_{n\to+\infty}\left|\sqrt{n}\mathbb{E}_{x,a}\left[u(X_n)\varphi(S_n)\right]-\frac{\nu(u)}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}}\varphi(y)e^{-(y-a)^2/2\sigma^2n}\mathrm{d}y\right|=0,$$

where the convergence is uniform in $(x,a)\in\mathbb{X}\times\mathbb{R}$.

We also need other elementary estimations whose proof is detailed.

Lemma 2.5 *There exist constants $c, C>0$ such that for every $x\in\mathbb{X}, a, b, \ell>0$ and $n\ge1$,*

$$\mathbb{P}_{x,a}(S_n\in[b,b+\ell])\le\frac{c}{\sqrt{n}}\,\ell. \tag{2.5}$$

and furthermore, for any $t > 0$, if $|a-b| > t\sqrt{n}$ then

$$\mathbb{P}_{x,a}(S_n \in [b, b+\ell]) \leq \frac{C}{\sqrt{n}}\ \ell\ e^{-ct^2}. \tag{2.6}$$

Proof. Assertion (2.5) is a consequence of Theorem 2.4 above. Assertion (2.6) is a more precise overestimation than (2.5) for large values of the starting point a, namely when $a \succeq \sqrt{n}$, as proved below.

We fix $h, t > 0$ and let $m := \lfloor n/2 \rfloor$ be the lower round of $n/2$. We decompose $\mathbb{P}_{x,a}(S_n \in [b, b+\ell])$ as follows.

$$\begin{aligned}\mathbb{P}_{x,a}(S_n \in [b, b+\ell]) &= \mathbb{P}_x(a + S_n \in [b, b+\ell]) \\ &= \underbrace{\mathbb{P}_x(a + S_n \in [b, b+\ell], |S_m| > t\sqrt{n}/2)}_{P_1(n,x,a,b,\ell)} + \underbrace{\mathbb{P}_x(a + S_n \in [b, b+\ell], |S_m| \leq t\sqrt{n}/2)}_{P_2(n,x,a,b,\ell)}.\end{aligned}$$

On the one hand, from the Markov property, inequality (2.5) and the central limit theorem for products of random matrices [14], there exists a strictly positive constant c such that, uniformly in x, a and b,

$$\begin{aligned}P_1(n,x,a,b,\ell) &= \int_{\mathbb{X}\times[-t\sqrt{n}/2, t\sqrt{n}/2]^c} \mathbb{P}_{x'}(a + a' + S_{n-m} \in [b, b+\ell]) \quad \mathbb{P}_x(X_m \in \mathrm{d}x', S_m \in \mathrm{d}a') \\ &\leq \frac{c\ \ell}{\sqrt{n-m}} \mathbb{P}_x(|S_m| > t\sqrt{n}/2) \\ &\preceq \frac{e^{-ct^2}}{\sqrt{n}}.\end{aligned}$$

On the other hand, when $|a-b| > t\sqrt{n}$, the conditions $|S_m| \leq t\sqrt{n}/2$ and $a + S_n \in [b, b+\ell]$ yield $|S_n - S_m| \geq t\sqrt{n}/2 - \ell$. Hence, for fixed h and n large enough,

$$\begin{aligned}P_2(n,x,a,b,\ell) &\leq \mathbb{P}_x(a + S_n \in [b, b+\ell], |S_n - S_m| > t\sqrt{n}/4) \\ &= \mathbb{P}_x(a + S_m + S_{n-m}\circ\theta^m \in [b, b+\ell], |S_{n-m}\circ\theta^m| > t\sqrt{n}/4) \\ &= \mathbb{P}_x(a + \ln|L_{m,1}X_0| + \ln|L_{n,m+1}X_m| \in [b, b+\ell], |L_{n,m+1}X_m| > t\sqrt{n}/4) \\ &\leq \mathbb{P}(a + \ln|L_{m,1}x| + \ln|L_{n,m+1}x'| \in [b-\Delta, b+\ell+\Delta], \\ &\qquad\qquad |\ln|L_{n,m+1}x'|| > t\sqrt{n}/4 - \Delta) \text{ for any } x' \in \mathbb{X}, \text{by Lemma 2.2} \\ &\leq \int_{\{|c| > t\sqrt{n}/4-\Delta\}} \underbrace{\mathbb{P}(a + \ln|L_{m,1}x| + c \in [b-\Delta, b+\ell+\Delta])}_{\leq \sup_{y\in\mathbb{X}, B\in\mathbb{R}} \mathbb{P}_y(S_m \in [B, b+\ell+2\Delta])} \mathbb{P}(\ln|L_{n,m+1}x'| \in \mathrm{d}c) \\ &\preceq \frac{1}{\sqrt{n}} \mathbb{P}(|\ln|L_{n,m+1}x'|| > t\sqrt{n}/4 - \Delta) \\ &\preceq \frac{e^{-ct^2}}{\sqrt{n}}\end{aligned}$$

uniformly in (x, a) by using again (2.5) and the central limit theorem for product of random matrices [14].

□

The next statement is analogous to the previous lemma when the walk $(a + S_n)_n$ is forced to remain positive up to time n.

Lemma 2.6 *There exists a constant $C > 0$ such that for all $x \in \mathbb{X}, a, b \geq 0, \ell > 0$ and $n \geq 1$,*

$$\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \quad \leq \quad C\ \frac{V(x,a)\ \ell}{n}. \tag{2.7}$$

Furthermore, there exists a constant $C > 0$ such that for any $\ell, t > 0, n \geq 1, a > \ell + 2\Delta + t\sqrt{n}$ and $b > \max\{t\sqrt{n}, \Delta\}$,

$$\mathbb{P}_{x,a}(\tau \leq n, S_n \in [b, b+\ell]) \leq C\frac{\ell}{\sqrt{n}} e^{-ct^2}. \tag{2.8}$$

Proof. For any $1 \leq m \leq n$,

$$\begin{aligned}
\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) &\leq \mathbb{P}_{x,a}(\tau > m, S_n \in [b, b+\ell]) \\
&= \int_{\mathbb{X}\times\mathbb{R}_+^*} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell])\ \mathbb{P}_{x,a}(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a') \\
&\preceq \frac{\mathbb{P}_{x,a}(\tau > m)}{\sqrt{n-m}} h \quad \text{by (2.5)} \\
&\leq c\ \frac{V(x,a)\ \ell}{n} \text{ by Proposition 2.1.}
\end{aligned}$$

To prove assertion (2.8), we work in two steps. Let $m = \lfloor n/2 \rfloor$.

Step 1. When $b > t\sqrt{n}$, by using the Markov property,

$$\begin{aligned}
&\mathbb{P}_{x,a}(\tau \leq m, S_n \in [b, b+\ell]) \\
&\quad = \sum_{k=1}^{m} \mathbb{P}_{x,a}(\tau = k, S_n \in [b, b+\ell]) \\
&\quad = \sum_{k=1}^{m} \int_{\mathbb{X}\times\mathbb{R}^-} \mathbb{P}_{x',a'}(S_{n-k} \in [b, b+\ell]) \quad \mathbb{P}_{x,a}(\tau = k, (X_k, S_k) \in \mathrm{d}x'\mathrm{d}a') \\
&\quad \leq \max_{n-m \leq \ell \leq n} \sup_{\substack{x' \in \mathbb{X} \\ |a'-b| > t\sqrt{n}}} \mathbb{P}_{x',a'}(S_\ell \in [b, b+\ell]) \sum_{k=1}^{m} \int_{\mathbb{X}\times\mathbb{R}^-} P_{x,a}(\tau = k, (X_k, S_k) \in \mathrm{d}x'\mathrm{d}a') \\
&\quad \leq \max_{n-m \leq \ell \leq n} \sup_{\substack{x' \in \mathbb{X} \\ |a'-b| > t\sqrt{n}}} \mathbb{P}_{x',a'}(S_\ell \in [b, b+\ell])\ \mathbb{P}_{x,a}(\tau \leq m) \\
&\quad \leq \max_{n-m \leq \ell \leq n} \sup_{\substack{x' \in \mathbb{X} \\ |a'-b| > t\sqrt{n}}} \mathbb{P}_{x',a'}(S_\ell \in [b, b+\ell]) \\
&\quad \preceq \frac{\ell}{\sqrt{n}}\ e^{-ct^2}, \quad \text{for some constant } c > 0, \text{ by (2.6).}
\end{aligned}$$

Step 2. We control the term $\mathbb{P}_{x,a}(m < \tau \leq n, S_n \in [b, b+\ell])$. By using the same argument to prove (2.3), it follows that

$$\begin{aligned}
&\mathbb{P}_{x,a}(m < \tau \leq n,\ S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}_x(\exists k \in \{m+1, \dots, n-1\} : a + S_k \leq 0, a + S_n \in [b, b+\ell]) \\
&\quad = \mathbb{P}(\exists k \in \{m+1, \dots, n-1\} : a + \ln|L_{n,1}x| - \ln|L_{n,k+1}X_k^x| \leq 0, a + \ln|L_{n,1}x| \in [b, b+\ell]) \\
&\quad \leq \mathbb{P}(\exists k \in \{m+1, \dots, n-1\} : b - \Delta - \ln|\tilde{y}L_{n,k+1}| \leq 0, a + \ln|\tilde{y}L_{n,1}| \in [b-\Delta, b+\ell+\Delta]) \\
&\quad = \mathbb{P}(\exists k \in \{m+1, \dots, n-1\} : b - \Delta - \ln|\tilde{y}R_{1,n-k}| \leq 0, a + \ln|\tilde{y}R_{1,n}| \in [b-\Delta, b+\ell+\Delta]) \\
&\quad \text{(by using again the fact that } (g_1, \dots, g_n) \text{ and } (g_n, \dots, g_1) \text{ have the same distribution)} \\
&\quad \leq \mathbb{P}(\exists \ell \in \{1, \dots, m\} : b - \Delta - \ln|\tilde{y}R_{1,\ell}| \leq 0, b - \Delta - \ln|\tilde{y}R_{1,n}| \in [a - \ell - 2\Delta, a]) \\
&\quad = \mathbb{P}_{\tilde{y}, b-\Delta}(\tilde{\tau} \leq m, \tilde{S}_n \in [a - \ell - 2\Delta, a]) \\
&\quad \preceq \frac{\ell}{\sqrt{n}} e^{-ct^2}, \quad \text{for some constant } c > 0,
\end{aligned}$$

where the last inequality is obtained by applying Step 1 above to the couple $(\tilde{\tau}, \tilde{S}_n)$ instead of (τ, S_n), assured by the condition $a - \ell - 2\Delta > t\sqrt{n}$ and $b > \Delta$.

□

3 Proof of Theorem 1.1

We adapt the proof of Theorem 5 in [5] and insist on the main differences. We fix two positive constants A and ϵ such that $A > 2\epsilon > 0$ and split $\mathbb{R}^+$ into three intervals : $]A\sqrt{n}; +\infty[$, $]0, 2\epsilon\sqrt{n}[$ and $I_{n,\epsilon,A} = [2\epsilon\sqrt{n}, A\sqrt{n}]$. The proof is decomposed into three steps.
Step 1.

$$\lim_{A \to +\infty} \limsup_{n \to +\infty} \left[n \sup_{\substack{x \in \mathbb{X} \\ b \geq A\sqrt{n}}} \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \right] = 0.$$

Step 2.

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \left[n \sup_{\substack{x \in \mathbb{X} \\ 0 < b \leq 2\epsilon\sqrt{n}}} \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \right] = 0.$$

Step 3. For any $A > 0$,

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \sup_{\substack{x \in \mathbb{X} \\ b \in I_{n,\epsilon,A}}} \left| n\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) - \frac{2}{\sigma\sqrt{2\pi n}} V(x,a)\ b\ \ell\ e^{-b^2/2n} \right| = 0.$$

Theorem 1.1 follows by combining these three steps; the convergence is obviously uniform over x.

We set $m = \lfloor n/2 \rfloor$.

Proof of Step 1. Let $a > 0$ and $b \geq A\sqrt{n}$. We rewrite $\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell])$ as $P_1 + P_2$, where

$$P_1 = \mathbb{P}_{x,a}(\tau > n, S_m \leq A\sqrt{m}, S_n \in [b, b+\ell])$$

and

$$P_2 = \mathbb{P}_{x,a}(\tau > n, S_m > A\sqrt{m}, S_n \in [b, b+\ell]).$$

By the Markov property, Proposition 2.1 and inequality (2.6), for some $c > 0$,

$$\begin{aligned}
P_1 &\le \mathbb{P}_{x,a}(\tau > m, S_m \le A\sqrt{m}, S_n \in [b, b+\ell]) \\
&\le \int_{\mathbb{X}\times]0, A\sqrt{m}]} \mathbb{P}_{x,a}(\tau > m, X_m \in \mathrm{d}x', S_m \in \mathrm{d}a')\ \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]) \\
&\le \mathbb{P}_{x,a}(\tau > m, S_m \le A\sqrt{m}) \sup_{\substack{x'\in\mathbb{X} \\ 0<a'\le A\sqrt{m}}} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]) \\
&\le \mathbb{P}_{x,a}(\tau > m) \sup_{\substack{x'\in\mathbb{X} \\ |b-a'|>A\sqrt{n}/4}} \mathbb{P}_{x',a'}(S_{n-m} \in [b, b+\ell]) \\
&\preceq \frac{V(x,a)}{\sqrt{n}} \times \frac{\ell}{\sqrt{n}}\ e^{-cA^2} \quad \preceq \quad \frac{V(x,a)}{n}\ \ell\ e^{-cA^2}. \qquad (3.1)
\end{aligned}$$

Similarly, by Proposition 2.1 and (2.5),

$$\begin{aligned}
P_2 &\le \mathbb{P}_{x,a}(\tau > m, S_m > A\sqrt{m}, S_n \in [b, b+\ell]) \\
&\le \mathbb{P}_{x,a}\left(S_m > A\sqrt{m} \middle| \tau > m\right)\ \mathbb{P}_{x,a}(\tau > m) \sup_{(x',a')\in\mathbb{X}\times\mathbb{R}_+^*} \mathbb{P}_{x',a'}\left(S_{n-m} \in [b, b+\ell]\right) \\
&\preceq \mathbb{P}_{x,a}\left(\frac{S_m}{\sigma\sqrt{m}} > \frac{A}{\sigma} \middle| \tau > m\right) \frac{V(x,a)}{\sqrt{m}} \frac{\ell}{\sqrt{n-m}} \\
&\preceq \frac{V(x,a)\ \ell}{n} \int_{A/\sigma}^{+\infty} t e^{-t^2/2} \mathrm{d}t \\
&= \frac{V(x,a)\ \ell}{n} e^{-A^2/2\sigma^2}. \qquad (3.2)
\end{aligned}$$

Hence, by combining (3.1) and (3.2), it follows that

$$\lim_{A\to+\infty} \limsup_{n\to+\infty} n\ \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \preceq \lim_{A\to+\infty} V(x,a)\ \ell\ \left(e^{-cA^2} + e^{-A^2/2\sigma^2}\right) = 0.$$

Proof of Step 2. Assume now $0 < b < 2\epsilon\sqrt{n}$. The Markov property and Proposition 2.1 yield

$$\begin{aligned}
\mathbb{P}_{x,a}(\tau > n,\ S_n \in [b, b+\ell]) & \\
&\le \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > n, S_m \in [i, i+1[, S_n \in [b, b+\ell]) \\
&\le \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > m, S_m \in [i, i+1[) \sup_{\substack{x'\in\mathbb{X} \\ a'\in[i,i+1[}} \mathbb{P}_{x',a'}(\tau > n-m, S_{n-m} \in [b, b+\ell]) \\
&\overset{\text{by (2.3)}}{\le} \sum_{i\in\mathbb{N}} \mathbb{P}_{x,a}(\tau > m, S_m \in [i, i+1[)\ \mathbb{P}_{\tilde{x}, b+\ell+\Delta}(\tilde{\tau} > n-m, \tilde{S}_{n-m} \in [i, i+\ell+2\Delta+1]) \\
&\overset{\text{by (2.7)}}{\le} C\frac{1+a}{m} \sum_{i\in\mathbb{N}} \mathbb{P}_{\tilde{x}, b+\ell+\Delta}(\tilde{\tau} > n-m, \tilde{S}_{n-m} \in [i, i+\ell+2\Delta+1]) \\
&\preceq \frac{V(x,a)}{n} \mathbb{P}_{\tilde{x}, b+\ell+\Delta}(\tilde{\tau} > n-m) \\
&\preceq \frac{V(x,a)}{n} \times \frac{1+b+\ell+\Delta}{\sqrt{n-m}} \quad \preceq \quad \frac{V(x,a)(1+2\epsilon\sqrt{n})}{n^{3/2}}.
\end{aligned}$$

We conclude the proof of Step 2 letting $n \to +\infty$, then $\epsilon \to 0$.

Proof of Step 3. We fix $b \in I_{n,\epsilon,A}$ and set $m_\epsilon = \lfloor \epsilon^3 n \rfloor$. We rewrite $\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell])$ as follows.

$$\begin{aligned}\mathbb{P}_{x,a}(\tau > n, &S_n \in [b, b+\ell]) \\ &= \underbrace{\mathbb{P}_{x,a}(\tau > n, |S_{n-m_\epsilon} - b| > \epsilon\sqrt{n}, S_n \in [b, b+\ell])}_{\Sigma_1(n,\epsilon)} \\ &\qquad + \underbrace{\mathbb{P}_{x,a}(\tau > n, |S_{n-m_\epsilon} - b| \le \epsilon\sqrt{n}, S_n \in [b, b+\ell])}_{\Sigma_2(n,\epsilon)}\end{aligned}$$

For $\Sigma_1(n, \epsilon)$, by the Markov property, Proposition 2.1 and (2.6), it follows that

$$\begin{aligned}&\Sigma_1(n, \epsilon) \\ &= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]^c} \mathbb{P}_{x',a'}(\tau > m_\epsilon, S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}\Big(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a'\Big) \\ &\le \sup_{\substack{x' \in \mathbb{X} \\ |a'-b| > \epsilon\sqrt{n}}} \mathbb{P}_{x',a'}(\tau > m_\epsilon, S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}(\tau > n - m_\epsilon, S_{n-m_\epsilon} \in [b - \epsilon\sqrt{n}, b+\epsilon\sqrt{n}]^c) \\ &\le \sup_{\substack{x' \in \mathbb{X} \\ |a'-b| > \frac{1}{\sqrt{\epsilon}}\sqrt{m_\epsilon}}} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}(\tau > n - m_\epsilon) \\ &\preceq \frac{h}{\sqrt{m_\epsilon}} e^{-c/\epsilon} \mathbb{P}_{x,a}(\tau > n - m_\epsilon) \\ &\preceq \frac{V(x,a)\ \ell}{n\epsilon^{3/2}\sqrt{1-\epsilon^3}} e^{-c/\epsilon}, \text{ uniformly in } b \in I_{n,\epsilon,A}.\end{aligned}$$

Therefore

$$\lim_{\epsilon\to 0} \limsup_{n\to+\infty} \sup_{\substack{x\in\mathbb{X} \\ b \in I_{n,\epsilon,A}}} |n\Sigma_1(n,\varepsilon)| \preceq \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{V(x,a)\ \ell}{n\epsilon^{3/2}\sqrt{1-\epsilon^3}} e^{-c/\epsilon} = 0. \tag{3.3}$$

For $\Sigma_2(n, \epsilon)$, similarly we obtain

$$\begin{aligned}&\Sigma_2(n, \epsilon) \\ &= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \mathbb{P}_{x',a'}(\tau > m_\epsilon, S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}\Big(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a'\Big) \\ &= \Sigma_2'(n,\epsilon) - \Sigma_2''(n,\epsilon), \end{aligned} \tag{3.4}$$

where

$$\Sigma_2'(n,\epsilon) := \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}\Big(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a'\Big)$$

and

$$\begin{aligned}&\Sigma_2''(n,\epsilon) \\ &:= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \mathbb{P}_{x',a'}(\tau \le m_\epsilon, S_{m_\epsilon} \in [b, b+\ell])\ \mathbb{P}_{x,a}\Big(\tau > n - m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a'\Big).\end{aligned}$$

We first treat the second term $\Sigma_2''(n,\epsilon)$. Since $b \geq 2\epsilon\sqrt{n}$, it holds that $a' \geq \epsilon\sqrt{n} \geq \sqrt{\frac{m_\epsilon}{\epsilon}}$ for any $a' \in [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]$. Hence, by (2.8) with $t = \frac{1}{2\sqrt{\epsilon}}$, for such a',

$$\mathbb{P}_{x',a'}(\tau \leq m_\epsilon, S_{m_\epsilon} \in [b, b+\ell]) \preceq m_\epsilon^{-1/2} e^{-c/\epsilon} = n^{-1/2}\epsilon^{-3/2}e^{-c/\epsilon}.$$

Consequently,

$$\begin{aligned}
\Sigma_2''(n,\epsilon) &\preceq n^{-1/2}\epsilon^{-3/2}e^{-c/\epsilon}\mathbb{P}_{x,a}(\tau > n-m_\epsilon, S_{n-m_\epsilon} \in [b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]) \\
&\leq n^{-1/2}\epsilon^{-3/2}e^{-c/\epsilon}\ \mathbb{P}_{x,a}(\tau > n-m_\epsilon) \\
&\preceq n^{-1/2}\epsilon^{-3/2}e^{-c/\epsilon}\frac{1+a}{\sqrt{n-m_\epsilon}} \quad \preceq \quad \frac{1+a}{n\sqrt{1-\epsilon^3}}\epsilon^{-3/2}e^{-c/\epsilon},
\end{aligned}$$

which implies that

$$\lim_{\epsilon\to 0}\limsup_{n\to+\infty}\sup_{\substack{x\in\mathbb{X}\\ b\in I_{n,\epsilon,A}}}|n\Sigma_2''(n,\varepsilon)| \preceq \lim_{\epsilon\to 0}\frac{1+a}{\sqrt{1-\epsilon^3}}\epsilon^{-3/2}e^{-c/\epsilon} \quad = 0. \tag{3.5}$$

It remains to control the term $\Sigma_2'(n,\epsilon)$. By Theorem 2.4, uniformly in $b \in I_{n,\epsilon,A}$,

$$\begin{aligned}
\Sigma_2'(n,\epsilon) &= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \mathbb{P}_{x',a'}(S_{m_\epsilon} \in [b,b+\ell])\mathbb{P}_{x,a}(\tau > n-m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a') \\
&= \int_{\mathbb{X}\times[b-\epsilon\sqrt{n}, b+\epsilon\sqrt{n}]} \frac{1}{\sigma\sqrt{2\pi m_\epsilon}}e^{-(b-a')^2/2\sigma^2 m_\epsilon}\ \ell\ (1+o_n(1)) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{x,a}(\tau > n-m_\epsilon, (X_{n-m_\epsilon}, S_{n-m_\epsilon}) \in \mathrm{d}x'\mathrm{d}a') \\
&\qquad\qquad\qquad\qquad\qquad (\text{with } o_n \text{ uniform in } b, a', \epsilon) \\
&= \frac{\ell}{\sigma\sqrt{2\pi m_\epsilon}}\ (1+o_n(1))\mathbb{E}_{x,a}\left[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon}; b-\epsilon\sqrt{n} \leq S_{n-m_\epsilon} \leq b+\epsilon\sqrt{n}; \tau > n-m_\epsilon\right] \\
&= \frac{1}{\sigma^2\pi}\ V(x,a)\ \frac{\ell}{\sqrt{m_\epsilon(n-m_\epsilon)}}\ (1+o_n(1)) \\
&\qquad \times \mathbb{E}_{x,a}\left[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon}1_{[b-\epsilon\sqrt{n},\ b+\epsilon\sqrt{n}]}(S_{n-m_\epsilon})/\tau > n-m_\epsilon\right].
\end{aligned} \tag{3.6}$$

The limit theorem for $(S_n)_n$ conditioned to stay in $\mathbb{R}^+$ (see Proposition 2.1) combined with the second Dini's theorem yields : for every fixed $\epsilon > 0$, as $n \to +\infty$,

$$\begin{aligned}
\sup_{(x,b)\in\mathbb{X}\times I_{n,\epsilon,A}} \Bigg|\mathbb{E}_{x,a}&\left[e^{-(b-S_{n-m_\epsilon})^2/2\sigma^2 m_\epsilon}1_{[b-\epsilon\sqrt{n},\ b+\epsilon\sqrt{n}]}(S_{n-m_\epsilon})/\tau > n-m_\epsilon\right] \\
&- \int_{|\sqrt{1-\epsilon^3}t-\frac{b}{\sqrt{n}}|<\epsilon} te^{-t^2/2}e^{-(b/\sqrt{n}-\sqrt{1-\epsilon^3}t)^2/2\epsilon^3}\mathrm{d}t\Bigg| \longrightarrow 0.
\end{aligned} \tag{3.7}$$

Since this last integral equals $\frac{b}{\sqrt{n}}e^{-b^2/2n}(2\pi\epsilon)^{3/2} + o(\epsilon^{3/2})$ (see [5] for the details), we obtain, combining (3.6) and (3.7),

$$\lim_{\epsilon\to 0}\limsup_{n\to+\infty}\sup_{(x,b)\in\mathbb{X}\times I_{n,\epsilon,A}}\left|n\Sigma_2'(n,\epsilon) - \frac{2\sqrt{2\pi}}{\sigma^2}\ V(x,a)\ \frac{b\ \ell}{\sqrt{n}}e^{-b^2/2n}\ \right| = 0. \tag{3.8}$$

The proof of Step 3 is complete by combining (3.3), (3.4), (3.5) and (3.8).

□

4 Proof of Theorem 1.2

Inequality (1.2) is proved in [19] Corollary 3.7. The proof of the lower bound (1.3) is based on Theorem 1.1 and is valid for $\ell > 2\Delta + 1$ and $b \geq \Delta$. As previously, we set $m = \lfloor n/2 \rfloor$. By the Markov property and (2.4),

$$
\begin{aligned}
\mathbb{P}_{x,a}(\tau > n, & S_n \in [b, b+\ell]) \\
& \geq \mathbb{P}_{x,a}(\tau > n, S_m \in [\sqrt{n}, \sqrt{2n}], S_n \in [b, b+\ell]) \\
& \geq \sum_{\substack{k \in \mathbb{N} \\ \sqrt{n} \leq k \leq \sqrt{2n}-1}} \mathbb{P}_{x,a}\Big(\tau > n, k \leq S_m \leq k+1[,\ b \leq S_m + S_{n-m} \circ \theta^m \leq b+\ell\Big) \\
& \geq \sum_{\substack{k \in \mathbb{N} \\ \sqrt{n} \leq k \leq \sqrt{2n}-1}} \int_{\mathbb{X} \times [k,k+1]} \mathbb{P}_{x,a}(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a') \\
& \qquad\qquad \mathbb{P}_{x',a'}(\tau > n-m, b \leq S_{n-m} \leq b+\ell) \\
& \geq \sum_{\substack{k \in \mathbb{N} \\ \sqrt{n} \leq k \leq \sqrt{2n}-1}} \int_{\mathbb{X} \times [k,k+1]} \mathbb{P}_{x,a}\Big(\tau > m, (X_m, S_m) \in \mathrm{d}x'\mathrm{d}a') \\
& \qquad\qquad \mathbb{P}_{\tilde{x},b-\Delta}(\tilde{\tau} > n-m, a' - \ell \leq \tilde{S}_{n-m} \leq a' - 2\Delta) \\
& \geq \sum_{\substack{k \in \mathbb{N} \\ \sqrt{n} \leq k \leq \sqrt{2n}-1}} \mathbb{P}_{x,a}\Big(\tau > m, k \leq S_m \leq k+1\Big) \\
& \qquad\qquad \mathbb{P}_{\tilde{x},b-\Delta}\Big(\tilde{\tau} > n-m, k+1-\ell \leq \tilde{S}_{n-m} \leq k - 2\Delta\Big) \qquad (4.1)
\end{aligned}
$$

By Theorem 1.1, there exists a constant $C_0 > 0$ such that for any $k \in \mathbb{N}$ satisfying $\sqrt{n} \leq k \leq \sqrt{2n} - 1$,

$$
\liminf_{n \to +\infty} n\mathbb{P}_{x,a}\Big(\tau > m, k \leq S_m \leq k+1\Big) \geq C_0
$$

and

$$
\liminf_{n \to +\infty} n\mathbb{P}_{\tilde{x},b-\Delta}\Big(\tilde{\tau} > n-m, k-1 \leq \tilde{S}_{n-m} \leq k - 2\Delta\Big) \geq C_0(\ell - 2\Delta - 1).
$$

Hence, inequality (4.1) yields, for n large enough,

$$
n^2 \mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \geq \frac{C_0^2}{2}(\sqrt{2n} - \sqrt{n})(\ell - 2\Delta - 1),
$$

which implies, for such n,

$$
\mathbb{P}_{x,a}(\tau > n, S_n \in [b, b+\ell]) \succeq \frac{\ell - 2\Delta - 1}{n^{3/2}}.
$$

This achieves the proof of inequality (1.3), taking for instance $\ell_0 = 4\Delta + 2$.

□

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